Acceleration, velocity and position in 1d

1. In the *Fellowship of the Ring*, Pippin Took drops a bucket down a well which takes a very long time to reach the "bottom." In the movie, this takes about 8 seconds.
    a. Assuming the bucket starts at rest at the top of the well and accelerates at a constant value of 10 m/s^2, how fast is the bucket going just before it hits bottom?
    b. How deep is the well? Ignore the time it takes for the sound to get back up the well.
    c. If we assume the speed of sound is 340 m/s, was it reasonable to ignore the time it took the sound to get back up the well?
    adapted and altered for our use from Understanding Physics (Cummings, Laws, Redish & Cooney; Wiley 2004)

What we DON'T want you doing is developing an algorithmic skillset which allows you to stop thinking so consider the following while trying to solve these problems (which are the sorts of questions that will allow you to set up the math and solve this numerically):

- What do we need to solve for? The initial or final velocity? The initial or final position?
- What is the initial velocity and initial position of the bucket? Is one or both of those arbitrary for your purposes?
- Which way do you want the vertical axis to point? Where should you put your origin?
- How precise should your answer be? (This will inform your answer for c.)

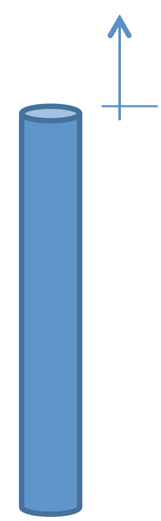
Here are two approaches, hopefully one answer! If you think this is long and drawn out, we're just covering all our bases. If you look up a "kinematics equation" and just plug and grind on HW or a test with far too little explanation, you will often be docked points.

First: how not to do it.

$$x = \frac{1}{2}at^2 = \frac{1}{2}(5)8^2 = 320$$

We don't know where you got this, we don't see units or signs, we don't see any kind of set up, reasoning or taking initial position or speed/velocity into consideration. Blech. The math is "right," but the brain may very well have been turned off the entire time. DEFINITELY POINTS OFF. On a test, we might not want as much as you see below, but justify and keep units around!

Second: probably the most natural way.



a. and b. Let's put the origin at the top of the well and the positive direction pointing "up." (Draw a picture like the one at right showing your choices and the well: it's worth 50 equations and 1000's of words.) *All* positions of this bucket then are non-positive in this problem. It starts at $x_{initial}$ = 0 and ends up at some negative depth x_{final}. Note that I don't put the minus sign there explicitly! I just know from the set up that x_{final} is some negative number. The problem states that the bucket starts from rest, so I don't have any leeway here for the initial speed: $v_{initial}$ = 0. Similarly, it's always falling, so I also know $v < 0$ for the rest of the trip down the well. Finally, while it's not stated explicitly that the acceleration is "down" (negative for our choice of origin and axis orientation), the fact that the bucket starts at rest but then keeps heading down means we also have no choice there either: $a < 0$, so, $a = -10$ m/s^2. We'll also choose to "start the clock" when the bucket first starts falling from the top of the well (a natural choice you probably don't have to justify in most problems) and that time marches forward positively (a sign convention we'd be utterly insane to mess with!). So, with subscript zero meaning $t = 0$ (initial), and with full vector notation (and keeping units when they show up!!!), we start with the acceleration and take indefinite integrals (antiderivatives, essentially):

$$\vec{a}_x = a_0\hat{\imath} = -10\ \mathrm{m/s^2}\hat{\imath} = \frac{d\vec{v}_x}{dt}$$

$$\vec{v}_x = (v_0 + a_0 t)\hat{\imath} = (0 - 10\ \mathrm{m/s^2}t)\hat{\imath} = -10\ \mathrm{m/s^2}t\hat{\imath} = \frac{d\vec{x}}{dt}$$

$$\vec{x} = (x_0 + v_0 t + a_0 t^2/2)\hat{\imath} = (0 + 0 - 5\ \mathrm{m/s^2} t^2)\hat{\imath} = -5\ \mathrm{m/s^2} t^2 \hat{\imath}$$

Because we keep integrating with an *indefinite* integral, we keep getting us a constant of integration ("+C" back in your calculus class) which we've labeled with malice aforethought as v_0 in one case and x_0 in the other). Solving for the final depth:

$$\vec{x}_{final} = x_{final}\hat{\imath} = -5\ \mathrm{m/s^2} t_{final}{}^2 \hat{\imath}$$
$$x_{final} = -5\ \mathrm{m/s^2}\ (8\ \mathrm{s})^2 = -320\ \mathrm{m}$$

The minus sign is relative to our choice of origin. What about the final speed before it hits the bottom?

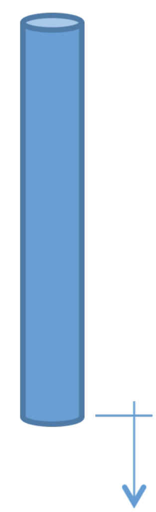
$$\vec{v}_{final} = v_{final}\hat{\imath} = -10\ \mathrm{m/s^2} t_{final}\hat{\imath}$$

$$v_{final} = -10\ \mathrm{m/s^2}\ (8\ \mathrm{s}) \ = -80\ \mathrm{m/s}$$

The minus sign here is the direction it was traveling (down). Did you see how we waited until the last line to 1) plug in numbers, 2) cancel unit vectors from both sides, and 3) kept units the whole time until the very end? Can you see what would change if we put the origin at the bottom of the well but kept it pointing "up" as positive (not much, just the value for the initial and final positions).

Third: let's make our lives difficult.

Just for kicks, let's redo this problem with the origin put at the *bottom* of the well and the positive direction pointing *down* (see arrow on drawing at right). Now the initial position is a negative number that we don't know yet and the **final** position is known to be zero. Furthermore the bucket is *always moving in the positive direction faster and faster so the acceleration is also positive!* The calculus is nearly identical but watch as the signs flip around and be careful about what you think is a variable (*t, x, v*) and what is a specific value of that variable ($x_{initial}$, v_{final}, etc.). Furthermore, let's claim that Gandalf starts timing this at $t_{initial}$ = 3 s (consequently it hits bottom eight seconds later at t_{final} = 11 s). You can now think of these as definite integrals, but of course, we'd better get the same answers, right?



$$\Delta\vec{v}_x \ = \int_3^{11} \vec{a}_x(t)\ dt = \int_3^{11} a_0\hat{\imath}\ dt = 10\ \mathrm{m/s^2} t\ \hat{\imath}]_3^{11} = 10\ \mathrm{m/s^2}(11\mathrm{s} - 3\mathrm{s})\hat{\imath} = 80\ \mathrm{m/s}\ \hat{\imath}$$

Notice the subtle difference here! The *definite* integral only tells me the total ***change*** in the speed from *t* = 3 s to *t* = 11 s. I now have to add on my given value for the initial velocity to determine the final velocity (it just happens to be a trivial step):

$$\Delta\vec{v} = \vec{v}_{final} - \vec{v}_{initial} = (v_{final}\hat{\imath} - 0\ \hat{\imath}) = 80\ \mathrm{m/s}\ \hat{\imath}$$

$$v_{final} = 80\ \mathrm{m/s}$$

Correctly positive since it's speeding down the well and we perversely chose that to be the positive direction. Now to get information on *x,* I need the full functionality of *v*(*t*) before I can integrate it to get delta x. Confused yet? Given the initial value and the time dependence:

$$\vec{v}_x(t) \ = (v_{initial} + a_0(t - t_{initial}))\hat{\imath} = (0 + 10\ \mathrm{m/s^2}(t - t_{initial}))\hat{\imath} = 10\ \mathrm{m/s^2}(t - t_{initial})\hat{\imath}$$

Note that *t* is a dependent variable but $t_{initial}$ is a constant, specific value (3 s). Even though I know it's 3 s, I will leave it explicitly as constant $t_{initial}$ until the bitter end: this is a good habit to follow!

$$\Delta\vec{x} = \int_3^{11} \vec{v}_x(t) dt = \int_3^{11} 10\ \mathrm{m/s^2}(t - t_{initial})\ dt\ \hat{\imath} = (10\ \mathrm{m/s^2} t^2)/2 - 10\ \mathrm{m/s^2} t_{initial} t]_3^{11}\hat{\imath}$$

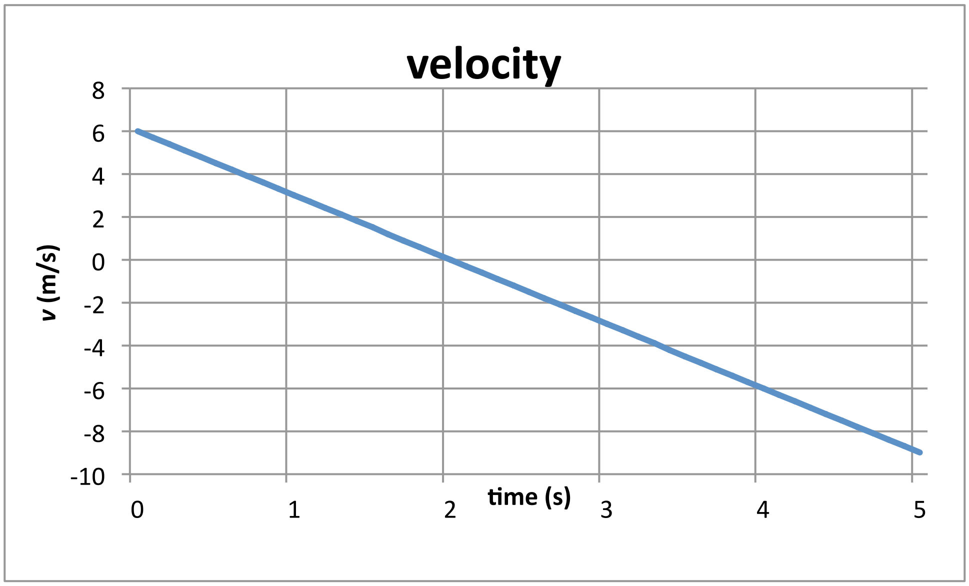
$$\Delta\vec{x} = [5\ \mathrm{m/s^2}(121\mathrm{s^2} - 9\mathrm{s^2}) - 10\ \mathrm{m/s^2}(3\mathrm{s})(11\mathrm{s} - 3\mathrm{s})]\ \hat{\imath} = 320\ \mathrm{m}\ \hat{\imath}$$

Pretty messy, eh? Again, that's just the *change* in *x!* Correctly a *positive* change, too, since downward is the positive direction! The *depth* of the well is 320 m, the final position is 0 so the initial position is:

$$\Delta\vec{x} = \vec{x}_{final} - \vec{x}_{initial} = (0\hat{\imath} - x_{initial}\,\hat{\imath}) = 320 \text{ m } \hat{\imath}$$
$$x_{initial} = -320 \text{ m}$$

c. For this, all we know is that the sound is traveling up the well very quickly (~3.4 football fields per second!). Is it accelerating at all? This isn't explicit, but since we are given a number without qualification ("the speed of sound is 340 m/s" doesn't specify initial or final so it must be either constant, or a useful average), we can assume the acceleration is zero and the time is just the distance divided by the speed. It takes about a second (since 320 m isn't much different from 340 m) which is therefore an error of ~12% in the timing (1/8). Backtracking, this means that the bucket must have only fallen for about 7 seconds, not 8 which works out to about 240 m (I'll let you do that math). Why such a big difference? Because of that quadratic behavior ($x \sim t^2$)!

2. Given this graph of the velocity of an object as a function of time, draw graphs of the position and acceleration as functions of time. Label the graphs as precisely as you can.



Here the problem is we don't know the initial position, so we can only graph the *change* in position. But since acceleration is the *change* in velocity, we can label the acceleration graph completely. It seems that the velocity is always decreasing at a constant rate – i.e., the change is always the same, so the acceleration is always the same, and in this particular case negative. In five seconds, the speed went from +6 to –9 m/s so the slope is simply:

$$a = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{(-9\text{m/s} - 6\text{m/s})}{(5\text{s} - 0\text{s})} = -3 \text{ m/s}^2$$

Of course we could have picked any two points for this calculation, but by picking the extremes, we minimize any error in eyeballing these values: can you see how?

For the position we need to think about the signs for a moment: the object starts out heading in the positive direction, slows down and then **reverses course** at 2 seconds and then starts heading in the negative direction. At 2 s, we say that the object is instantaneously at $v = 0$. This is very much like throwing your keys up in the air and considering the values of v and a at the "top:" the keys instantaneously stop and come back down, but the acceleration is always there (constant and in the down direction). We don't know the initial position, so we can either set it equal to zero (arbitrary) or claim we can only graph Δx, which is strictly true (and the "initial" Δx is meaningless since nothing's changed yet! $\Delta x_{initial} = \left(x_{final} - x_{initial}\right)_{initial} = x_{initial} - x_{initial} = 0$).

The equation for v is (by inspection, the vertical intercept is 6 m/s) $v(t) = v_0 + at = 6 \text{ m/s} - 3 \text{ m/s}^2 t$. Now you can integrate one timeslice at a time if you like to figure out each new value of delta x, or you can figure out the equation in time for the whole delta x by taking the antiderivative and move x_0 to the "other side" of the equal sign:

$$x(t) = x_0 + v_0 t + a_0 t^2/2 = x_0 + 6\ \mathrm{m/s}\ t - 1.5\ \mathrm{m/s^2}\ t^2$$

$$\Delta x(t) = x(t) - x_0 = 6\ \mathrm{m/s}\ t - 1.5\ \mathrm{m/s^2}\ t^2$$

I’ve lined up all three graphs together below: you should try lining up the behavior (between position change and velocity especially!) for each moment in time vertically!